Statistical thermodynamics of a two-dimensional relativistic gas

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In this paper we study a fully relativistic model of a two-dimensional hard-disk gas. This model avoids the general problems associated with relativistic particle collisions and is therefore an ideal system to study relativistic effects in statistical thermodynamics. We study this model using molecular-dynamics simulation, concentrating on the velocity distribution functions. We obtain results for x and y components of velocity in the rest frame (Γ) as well as the moving frame (Γ'). Our results confirm that Jüttner distribution is the correct generalization of Maxwell-Boltzmann distribution. We obtain the same "temperature" parameter β for both frames consistent with a recent study of a limited one-dimensional model. We also address the controversial topic of temperature transformation. We show that while local thermal equilibrium holds in the moving frame, relying on statistical methods such as distribution functions or equipartition theorem are ultimately inconclusive in deciding on a correct temperature transformation law (if any).

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I. INTRODUCTION

The question of how thermodynamic properties transform in a moving coordinate system were raised soon after Einstein's fundamental paper in 1905 [1]. In no more than half a century the introduction of several relativistically consistent generalization of thermodynamics led to such a confusing atmosphere in which one could not decide whether a moving body appears cooler, hotter, or at the same temperature as the body at rest. The most cited view is presented by Planck [2] and Einstein [3], who believed that temperature of a moving body would be Lorentz contracted. A different view was proposed later by some authors notably Ott [4] and Arzeliés [5], suggesting that a body in motion would appear relatively hot. Finally, in 1966 Landsberg [6,7] put forth the third suggestion, namely, the Lorentz-invariant temperature view. However, 30 years later Landsberg and Matsas [8,9] and recently Sewell [10] proposed another view, that of nonexistence of universal Lorentz transformation of temperature that further intensified the controversies over the subject.

Since its early days relativistic thermodynamics has changed from a theoretically interesting problem to a practically important subject due to its application in the proper interpretation of experiments in high energy and astrophysics [11–13]. Nevertheless, there is still no consensus on many features of this theory. One reason for the ongoing discussion is the lack of experimental evidences or numerical investigations. Among few exceptions is an interesting paper by Cubero *et al.* [14] who have shown that a simple one-dimensional model of relativistic dynamics favors Jüttner distribution function [15] as the correct generalization of Maxwell-Boltzmann (MB) distribution,

$$f_J(\mathbf{v}) = m^d \gamma(\mathbf{v})^{2+d} \exp[-\beta_J m\gamma(\mathbf{v})]/Z_J, \tag{1}$$

where d is dimension, Z_J is normalization constant, $E = m\gamma(\mathbf{v})$ is relativistic energy, and $\gamma(\mathbf{v}) = (1-v^2)^{-1/2}$ is the Lorentz factor in natural units with speed of light $c=1$.

Although the model used in [14] is one dimensional and lacks many features of a real physical system, it provides strong evidence against other generalizations of the Maxwellian, especially the "modified" Jüttner function [16–18],

$$f_{\mathrm{MJ}}(\mathbf{v}) = \frac{m^d}{Z_{\mathrm{MJ}}} \frac{\gamma(\mathbf{v})^{2+d}}{m\gamma(\mathbf{v})} \exp[-\beta_{\mathrm{MJ}} m\gamma(\mathbf{v})], \tag{2}$$

Jüttner distribution can be used as the cornerstone of our understanding of relativistic statistical mechanics in the same manner that MB distribution illuminates the underlying microscopic roots of classical thermodynamics. The most challenging step, however, is defining a proper thermometer in order to relate the Lagrange multiplier β to the temperature of the system. This problem is mostly treated as trivial in the literature but one should note that, the correct transformation of temperature, like any other quantity, depends crucially on the practical methods we implement for its measurement.

Here, we model a two-dimensional (2D) gaseous system with realistic features which at the same time allows for implementation of full relativistic dynamics. Since this model is both realistic and fully relativistic, it can be used as an ideal numerical laboratory in order to investigate many issues concerning relativistic generalization of statistical thermodynamics. Using standard relativistic transformations, we obtain directional distribution functions for both the rest as well as the moving frame. We study these functions numerically using molecular-dynamics simulations of our 2D model. Our results indicate that Jüttner distribution is the correct generalization. We also show that the same temperature parameter is obtained in both frames. Finally we discuss the implication of our results for a proper temperature transformation. In this regard, while verifying the important concept of local thermal equilibrium, we argue that these methods are ultimately inconclusive on deciding a correct temperature transformation law.

II. MODEL

We propose to study an idealized two-dimensional system of impenetrable hard disks with purely repulsive binary interaction $U(r)$,

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$$U(r) = \begin{cases} +\infty, & r < \sigma \\ 0, & r \geq \sigma. \end{cases} \quad (3)$$

The disklike particles move in straight lines at constant speed and change their momenta instantaneously when they touch at distance σ [19]. Hence, in order to simulate the dynamics, we must find the next collision and compute the changes in momenta of the colliding pair, considering the relativistic laws of conservation of energy and momentum in two-dimensional space. In order to solve this problem exactly, we add the assumption that when two hard disks collide, the force is exerted along the line connecting their centers, $\mathbf{r}_{ij} = \mathbf{r}_i - \mathbf{r}_j$. Therefore, the components of momenta perpendicular to $\mathbf{r}_{ij}$ remain unchanged ($\hat{p}_{i,\perp} = p_{i,\perp}$) and the parallel components change in the same way as the one-dimensional case [20],

$$\hat{p}_{i,\parallel} = \gamma(v_{cm})^2[2v_{cm}E_i - (1+v_{cm}^2)p_{i,\parallel}],$$

$$\hat{E}_i = \gamma(v_{cm})^2[(1+v_{cm}^2)E_i - 2v_{cm}p_{i,\parallel}], \quad (4)$$

where hatted quantities refer to momenta after collision and $v_{cm} = (p_{i,\parallel} + p_{j,\parallel})/(E_i + E_j)$ is the collision invariant, relativistic center-of-mass velocity of the two particles. With the same rules for particle j, a deterministic, time-reversible canonical transformation at each collision is defined. The additional assumption means that particles do not slide on each other when they collide. In contrast to the one-dimensional model, such elastic binary collisions lead to equilibrium even if colliding particles carry the same rest masses [14]. In our simulation we have used N particles of equal rest masses m that are constrained to move in a square box of linear size L. We use periodic boundary condition. Note that in order to simulate a stationary system in the rest frame, the center-of-mass momentum must be put to zero manually. This condition would automatically be satisfied (if not at each instant but at least on time average) if fixed reflecting walls were used [21].

III. RESULTS

A. Rest frame

In order to obtain the equilibrium state of the system we let the two-dimensional gas equilibrate (typically after 10^2N collisions) and measure velocities of particles at equal times with respect to laboratory frame. To collect more data, we repeated this procedure every $10N$ collisions. Simulation results for $N=100$ particles are presented. Particles are initially placed on a square lattice of constant $L/\sqrt{N}$ and velocities are chosen randomly in δ vicinity (with δ a small number) of $|\mathbf{v}_0| = \sqrt{1-(1/\eta)^2}$, corresponding to mean energy per particle $\epsilon = \eta m$. As $\eta \to 1$, the kinetic energy becomes small compared to the rest mass energy, recovering the dynamics of (classical) nonrelativistic model. The $\eta \to \infty$ limit, on the other hand, corresponds to a highly relativistic model.

Theoretically, the $x(y)$ component of Jüttner and modified Jüttner velocity distribution is obtained by integrating Eqs. (1) and (2) over $v_y(v_x)$,

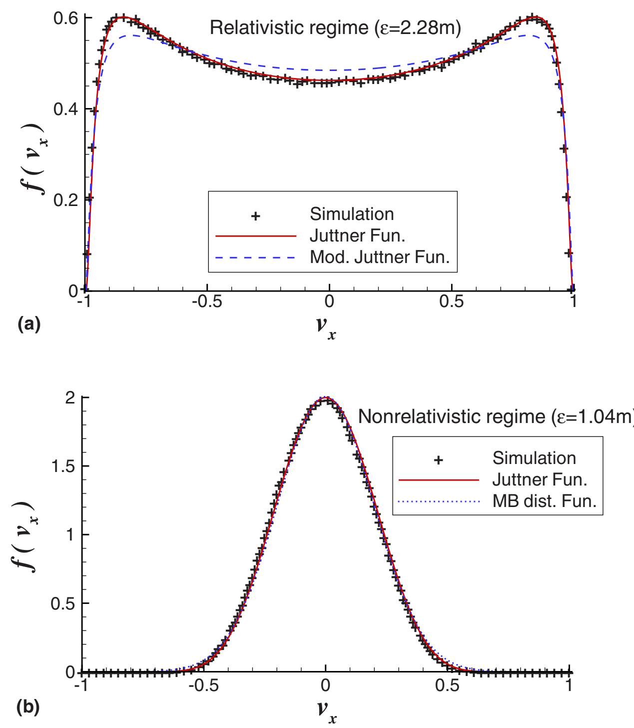

FIG. 1. (Color online) Equilibrium velocity distributions in the rest frame Γ: numerically obtained x component of single-particle velocity distribution (+) from simulation of $N=100$ particles of mass $m=0.1$. (a) Here, $\epsilon=2.28m$ and the corresponding temperature parameters are $\beta_J=11.4$, $\beta_{MJ}=7.8$. (b) $\epsilon=1.04\ m$, $\beta_J=253.1$, and $\beta_{MB}=259.8$. A significant deviation from modified Jüttner function is evident in the relativistic regime. Similar results are obtained for $f(v_y)$.

$$f_J(v_x) = \frac{m^2}{Z_J}\gamma(v_x)^3[K_2(\beta_J m\gamma(v_x)) + K_0(\beta_J m\gamma(v_x))], \quad (5)$$

$$f_{MJ}(v_x) = \frac{2m}{Z_{MJ}}\gamma(v_x)^2 K_1(\beta_{MJ} m\gamma(v_x)), \quad (6)$$

with K_n denoting modified Bessel functions of the second kind [22]. Here, the parameter $\beta_{J/MJ}$ is determined by means of the following procedure: we have, for the average energy,

$$\epsilon = E_{tot}/N = \int_{|\mathbf{v}|<1} d^d\mathbf{v} f(\mathbf{v}) m\gamma(\mathbf{v}). \quad (7)$$

Computing the right hand side (rhs) of Eq. (7) for Jüttner and modified Jüttner in the two-dimensional case gives $\text{rhs}_J = (\beta^2m^2+2\beta m+2)/(\beta(\beta m+1))$ and $\text{rhs}_{MJ} = 1/\beta + m$, respectively. By inserting E_{tot}, N, and m into these equations, the parameter $\beta_{J/MJ}$ consistent with Jüttner and modified Jüttner velocity distribution is uniquely determined.

We now check these results by considering two cases with $\eta=1.04$ and $\eta=2.28$. As shown in Fig. 1, the obtained single-particle distribution of velocity x-component (+) agrees with Jüttner function (solid lines) in both regimes and converges to MB distribution in the nonrelativistic limit. A significant deviation from modified Jüttner distribution (dashed lines) is also evident. Exact same diagrams are ob-

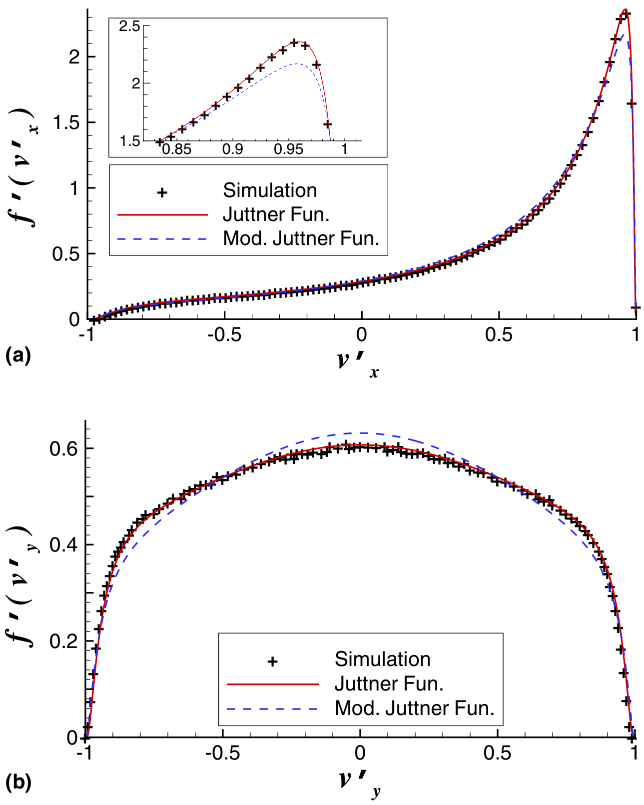

FIG. 2. (Color online) Equilibrium velocity distributions in the moving frame Γ' with relative velocity $u=0.5$. The system parameters are the same as Fig. 1(a) in particular $\beta_J=11.4$ and $\beta_{MJ}=7.8$. Note the breaking of symmetry in part (a) where the inset shows more details of the peak.

tained for y component of velocity (not shown). In particular, the y-component results were fitted with the same parameter β as the x-component data. This shows that our system has equilibrated properly through successive collisions.

B. Moving frame

We now turn to the more interesting question of equilibrium velocity distribution of a relativistic gas in motion. For this, we examine the system from the point of view of an observer who sees that the system's frame, Γ', is moving with a uniform velocity u in x direction with respect to his rest frame, Γ. Using the entropy maximization principle, the single-particle distribution will be determined by an additional constraint on the system, namely, that of a definite total momentum $\mathbf{p}'$ [23],

$$f'_J(\mathbf{v}') = \frac{m^d \gamma(\mathbf{v}')^{d+2}}{\gamma(u) Z_J} \exp[\beta_J \gamma(u) m \gamma(\mathbf{v}')(1-\mathbf{u}\cdot\mathbf{v}')], \quad (8)$$

$$f'_{MJ}(\mathbf{v}') = \frac{m^d \gamma(\mathbf{v}')^{d+2}}{\gamma(u) Z_{MJ}} \frac{\exp[\beta_{MJ}\gamma(u) m\gamma(\mathbf{v}')(1-\mathbf{u}\cdot\mathbf{v}')]}{\gamma(u) m \gamma(\mathbf{v}')(1-\mathbf{u}\cdot\mathbf{v}')}. \quad (9)$$

The primed quantities are measured in the moving frame and the additional $\gamma(\mathbf{u})$ term in the denominators is due to the contraction of the moving box that encloses the system [24]. Figure 2 shows the results for a system similar to Fig. 1 with $u=0.5$. Note that here, $x(y)$ component of velocities are measured Γ' simultaneously. The solid and dashed lines are the velocity $x(y)$ component of velocity distribution obtained, respectively, by integrating Eqs. (8) and (9) over $v_y(v_x)$, e.g.,

$$f'_J(v'_x) = \frac{m^2}{Z_J}\gamma(v'_x)^3[K_2(\beta m\gamma(u)\gamma(v'_x)(1-uv'_x)) + K_0(\beta m\gamma(u)\gamma(v'_x)(1-uv'_x))], \quad (10)$$

Other components of these distributions cannot be obtained in closed form and are therefore plotted in Fig. 2 using numerical integration. The parameter β used to fit the data in Fig. 2 is the same as that of the rest frame. Note that this parameter is obtained exactly as a function of system parameters (m,ϵ) in Γ. However, in Γ', β is a fitting parameter which turns out to be the same as that in Γ.

Therefore as clearly seen from numerical results, our two-dimensional model shows Jüttner distribution as the correct relativistic version of MB distribution. The fact that the same parameter β is obtained from both x and y component velocities shows equilibration. However, more importantly, the fact that same β is obtained from both Γ and Γ' frames seems to indicate the invariance of temperature consistent with earlier work of Landsberg [6] and previous simulation results [14]. We now discuss if this agreement can shed light on the long-lasting question of how temperature transforms in a moving frame.

IV. PROPER THERMOMETER

A commonly used definition of equilibrium temperature in the literature is $T=(k_B\beta)^{-1}$, where β is the Lagrange multiplier emerging in the velocity distribution function. One may use this definition and the equality of parameter β in moving and rest frame (Fig. 2) to deduce that temperature is Lorentz invariant, $T'=T$ [14]. However, there is another point of view which is also consistent with our results. Comparing Eq. (8) with the general form of the distribution function as $f\propto e^{-(\alpha+\beta\varepsilon+\omega\cdot\mathbf{P})}$, one is led to believe that the Lagrange multiplier is $\gamma(u)\beta$ which gives $T'=T/\gamma(u)$ [2,3,23].

To see this point better, note that the above definition of temperature has its roots in the equipartition theorem as well as basic thermodynamic relation, $dE=TdS-PdV+\mu dN$. The relativistic version of these methods is widely used in order to find the temperature transformation law [2,4,5,14,23,25]. However, despite the popularity of these approaches, they are not decisive either [26–28]. To illuminate, consider the Lorentz-invariant equipartition theorem for a system moving with velocity u parallel to x axis [7],

$$\left\langle\left\langle \frac{p'^2_{ix}}{m'_i} - up'_{ix}\right\rangle\right\rangle = \left\langle\left\langle \frac{p'^2_{iy}}{m'_i}\right\rangle\right\rangle = \frac{1}{\beta\gamma(u)}, \quad (11)$$

where the primed quantities are measured in the moving frame [e.g., $m'_i=\gamma(\mathbf{v}_i)m_i$] and averages $\langle\langle\cdots\rangle\rangle$ are taken Γ' simultaneously. One may apply either hypothesis that $\langle\langle p'^2_{ix}/m'_i - up'_{ix}\rangle\rangle$ or $\langle\langle \gamma(u)(p'^2_{ix}/m'_i - up'_{ix})\rangle\rangle$ are the statistical

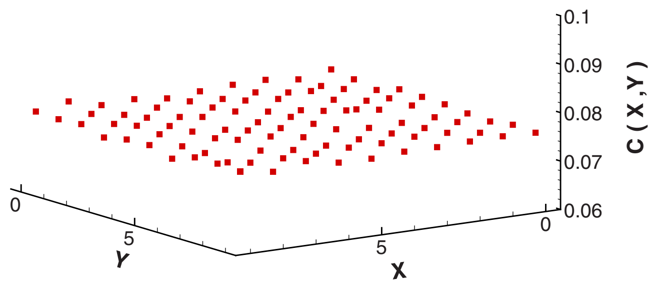

FIG. 3. (Color online) Local thermal equilibrium in the box as seen by a moving observer: The profile $\mathcal{C}(X,Y)$ is measured Γ' simultaneously for the same system as in Fig. 2 with L_x and L_y, each divided into ten parts. The expected value of $[\beta\gamma(u)]^{-1}$ is 0.076 which agrees well with the obtained value of $\mathcal{C}$ throughout the lattice.

thermometer of the moving system and find it compatible with the generalized theorem. Thus, as Landsberg has mentioned in [7]: "the argument from equipartition does not enable one to discriminate on theoretical grounds between accepted theory and Lorentz-invariant temperature."

Furthermore, in some recent papers it is claimed that there exists no universal and continuous Lorentz transformation of temperature at all [8,9]. The argument is based on the fact that black body radiation of a moving body is direction dependent. Therefore, a bath which is thermal in an inertial frame is nonthermal in a moving frame. Does local thermal equilibrium (LTE) hold in our system? To check, we divide the box into n cells of area $\Delta A' = (L'_x L'_Y)/n^2$ and calculate the quantity $\mathcal{C} = \langle\langle p'^2_{ix}/m'_i - up'_{ix}\rangle\rangle$, which we consider to be proportional to temperature, in each cell. The numerical result shown in Fig. 3 indicates that each cell, as seen by a moving observer, is characterized by a constant value which coincides with the value of $[\beta\gamma(u)]^{-1}$. This indicates that LTE, which is the necessary condition to introduce a well defined temperature, is fulfilled at least for our model.

V. CONCLUDING REMARKS

It seems that the longstanding issue of relativistic thermodynamics is related to the longstanding issue of irreversibility in thermodynamics. The lack of consensus on these issues is related to the lack of concise mapping between dynamical description of a system on one hand and a thermodynamic description on the other. We cannot define temperature (or entropy) as an exact function of dynamical variables. In this work we have modeled a useful, realistic system of a relativistic gas which overcomes the difficulties associated with implementation of particle interactions in a relativistically consistent manner [14,29]. We have shown that Jüttner function is the correct velocity distribution function in both rest and moving frames, with components either along or perpendicular to the relative velocity u. Furthermore, our results indicate that, with a certain definition of statistical thermometer, one can choose $\beta' = \beta$, i.e., a Lorentz-invariant temperature, without running into inconsistencies. However, $\beta' = \gamma(u)\beta$ could just as well be argued to be a valid choice, depending on a choice of thermometer. Such inconclusiveness inherit in statistical analysis like ours leads one to consider a covariant formulation of thermodynamics where temperature is generalized to a tensorial quantity whose transformation is no longer an issue [26,30]. In this view thermodynamic temperature is considered as a *proper* feature of a thermodynamic system, much like mass in relativistic mechanics [31].

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